Stabilizing Multi-agent Zero-sum Games with Spectral Damping by Optimistic GDA

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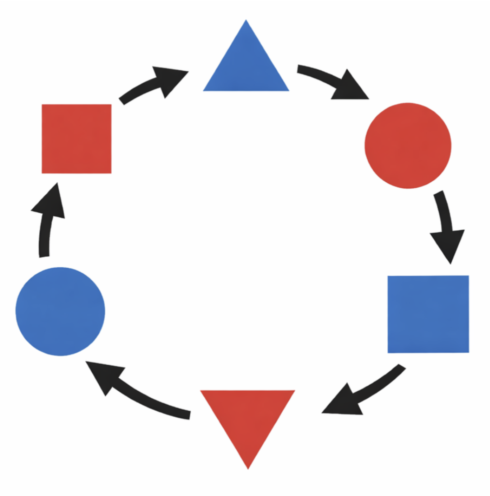
Motivation: Cycling in Adversarial Games

- Cyclic behavior appear in multi-agent adversarial games. This is a major limitation of Gradient Descent Ascent when solving saddle point (Nash Equilibriums) problems.
- OGDA/ OMD can solve oscillation with distributed agent-wise computation.
- More flexible design then uniform optimism parameter through agents for further optimization as well as implementation under certain constraints.



Problem Studied: Bilinear Multi-agent Game

- N agents. Scalar strategy state.

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathcal{A}^n := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \subseteq \mathbb{R}^n.$$

- Skew symmetric interaction matrix D (zero-sum).

$$D \in \mathbb{R}^{n\times n} \qquad D^\top = -D.$$

- Bilinear Utility

$$u_i(z) = z_i (Dz)_i, \qquad U(z) = \begin{bmatrix} u_1(z) \\ u_2(z) \\ \vdots \\ u_n(z) \end{bmatrix} = \operatorname{diag}(z) D z,$$

Method: OGDA with matrx-valued Optimism

- Standard OGDA step update for each agent:

$$z_i^{(t+1)} = z_i^{(t)} + \eta (Dz^{(t)})_i + \eta \frac{\partial u_i}{\partial z_i}^{(t)} - \frac{\partial u_i}{\partial z_i}^{(t-1)},$$

- Concatenated step update for OGDA with matrix valued optimism (M):

$$z^{(t+1)} = z^{(t)} + \eta D z^{(t)} + \eta M D \left(z^{(t)} - z^{(t-1)} \right)$$

Theoretical Result: Sufficient Bound for Convergence

- We find a sufficient bound for convergence for the system. The left-hand side depends on the spectral properties of optimism matrix M and interaction matrix D. The right-hand side can be bounded by norm.

$$\begin{aligned}
\Re(u_i^* M u_i) > \; & \underbrace{\frac{1}{2}}_{\eta^0} \\
& + \eta \left[\underbrace{\frac{5}{4|\omega_i|^2} d^3 m^2 + \frac{25}{16|\omega_i|^2} d^3 m}_{\textit{from } \frac{\eta}{|\omega_i|^2} S} + \underbrace{\frac{4a^2}{\delta|\omega_i|^2}}_{\textit{from } \frac{\eta}{|\omega_i|^2} \frac{4}{\delta} a^2} + \underbrace{|\omega_i|\,|M|}_{\textit{given}} \right] \\
& + \eta^2 \left[\underbrace{\frac{1}{2}|\omega_i|^2 m^2}_{\textit{given}} + \underbrace{\frac{10a}{\delta|\omega_i|^2} d^3 m^2 + \frac{25a}{2\delta|\omega_i|^2} d^3 m}_{\textit{from } \frac{\eta}{|\omega_i|^2} \frac{8a}{\delta} \eta S} \right] \\
& + \eta^3 \left[\underbrace{\frac{25}{4\delta|\omega_i|^2} d^6 m^4 + \frac{125}{8\delta|\omega_i|^2} d^6 m^3 + \frac{625}{64\delta|\omega_i|^2} d^6 m^2}_{\textit{from } \frac{\eta}{|\omega_i|^2} \frac{4}{\delta} \eta^2 S^2} \right].
\end{aligned} \tag{22}$$

Experiment: Setup

- N=8, eta=0.01.
- The skew-symmetric interaction matrix D is built from an all-ones strict upper triangle with two modified entries, D_{03} = 2 and D_{47} = 1.5.
- The tested M are as follows:

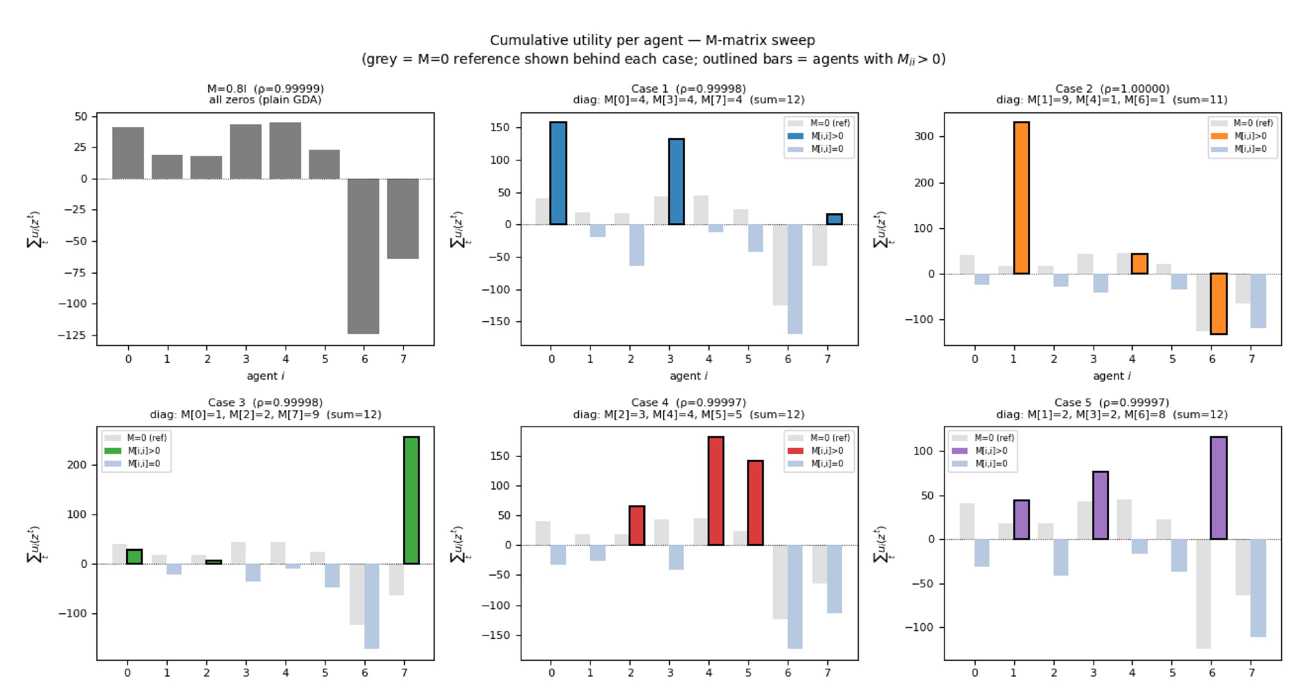
M-MATRIX SWEEP: SPECTRAL RADIUS AND FINAL INFINITY NORM ($n = 8$, $\eta = 0.01$, $T = 2000$, NOISELESS).

Case	Non-zero diagonals of M	$\rho(T(\eta))$	$\lVert z^T \rVert_\infty$
$M = 0.8I$	uniform	0.999991	1.094
Case 1	$M_{00} = 4,\ M_{33} = 4,\ M_{77} = 4$	0.999983	0.609
Case 2	$M_{11} = 9,\ M_{44} = 1,\ M_{66} = 1$	1.000000†	1.083
Case 3	$M_{00} = 1,\ M_{22} = 2,\ M_{77} = 9$	0.999982	0.690
Case 4	$M_{22} = 3,\ M_{44} = 4,\ M_{55} = 5$	0.999972	0.861
Case 5	$M_{11} = 2,\ M_{33} = 2,\ M_{66} = 8$	0.999973	0.936

†Marginal stability to six decimal places.

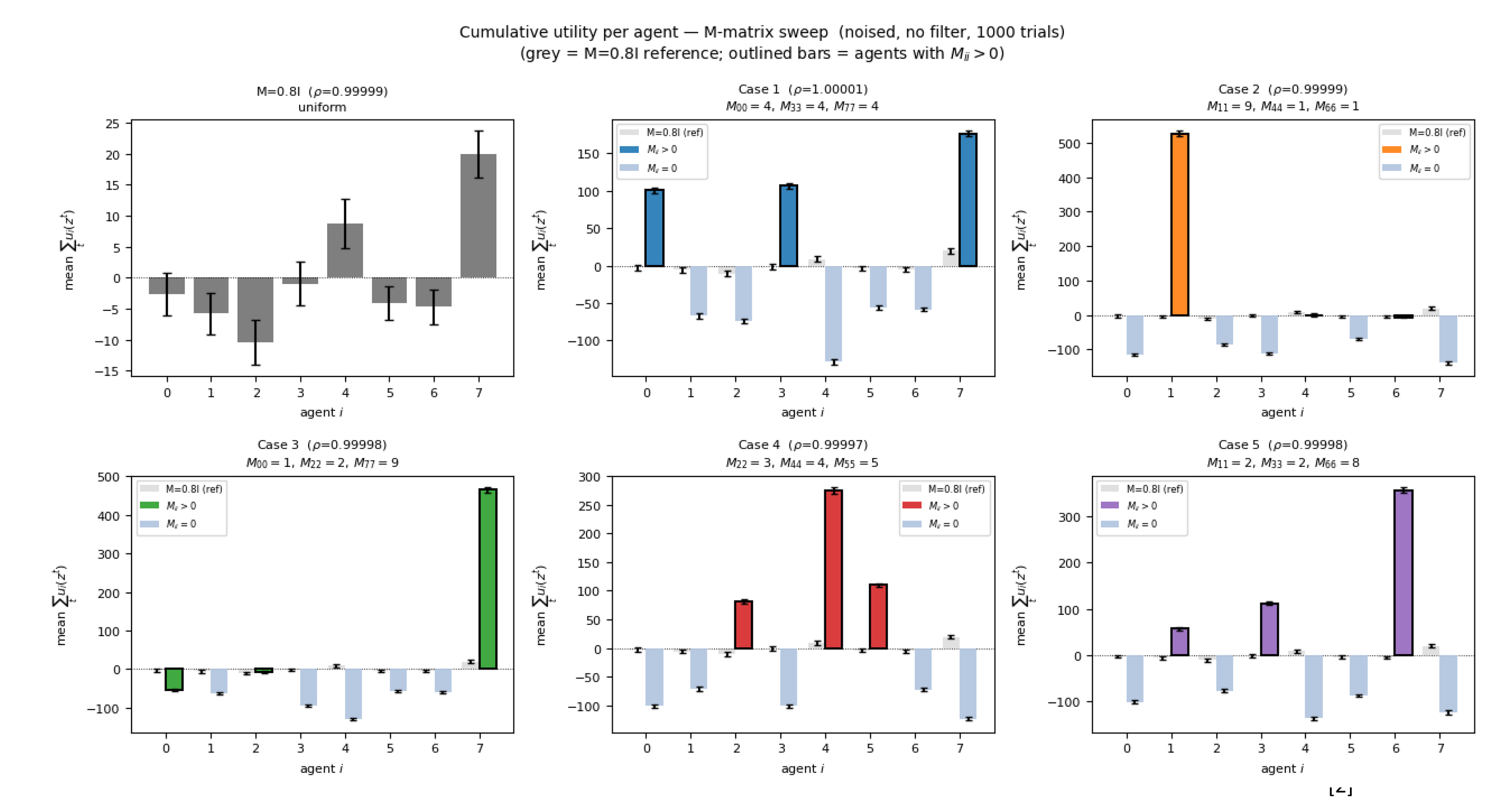
Experiment: No Noise

- Cumulative utility of agents. Highlight bare corresponds to agents with m_i>0



Experiment: Noised

- Cumulative utility of agents. Highlight bare corresponds to agents with m_i>0



Conclusion

- Certain multi-agent systems under pure zero-sum adversarial setting can achieve convergence with spectral damping.
- The boundary offers a much more flexible algorithm design, including partial damping or cooperation in small groups
- Contributing to stability of the overall systems benefit individuals in terms of cumulative utility.

Potential Applications, Future Work and References

- Our model gives insight about designing algorithms under adversarial settings with partial damping. It has the potential of being integrated into more complicated algorithms that suffer from cyclic behavior in adversarial learning.
- Our Experiment gives insight about how part of the agents can help regularize a game in pure zero-sum setting. Contributing to stability of the system may bring profit for agents themselves.
- More work can be done in Stochastic analysis.
- More work in cooperation in subset of agents, and fully explore the potential of the optimism matrix design.

REFERENCES

[1] C. Daskalakis, A. Ilyas, V. Syrgkanis, and H. Zeng, "Training GANs with optimism," in *International Conference on Learning Representations (ICLR)*, 2018.

[2] F. Schäfer and A. Anandkumar, "Competitive gradient descent," in *Advances in Neural Information Processing Systems*, 2019.

[3] P. Mertikopoulos, E. V. Belmega, and A. N. Moustakas, "Optimistic mirror descent in saddle-point problems: Going the extra (gradient) mile," *SIAM Journal on Optimization*, vol. 29, no. 3, pp. 2077–2102, 2019.

[4] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[5] K. J. Arrow, L. Hurwicz, and H. Uzawa, *Studies in Linear and Non-Linear Programming*. Stanford University Press, 1958.

[6] J. B. Rosen, "Existence and uniqueness of equilibrium points for concave n-person games," *Econometrica*, vol. 33, no. 3, pp. 520–534, 1965.

[7] P. Mertikopoulos *et al.*, "Cycles in adversarial learning," *SIAM Journal on Optimization*, 2018.

[8] V. M. Popov, "Hyperstability of control systems," *Springer*, 1980.

[9] G. Gidel *et al.*, "A variational inequality perspective on generative adversarial networks," *ICLR*, 2019.

[10] P. Bailey and G. Piliouras, "Finite-time analysis of two-time-scale gradient descent-ascent," *NeurIPS*, 2020.

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 2012.

[12] T. Kato, *Perturbation theory for linear operators*. Springer, 1995.

[13] A. Fallah, A. Ozdaglar, and S. Pattathil, "An optimal multistage stochastic gradient method for minimax problems," 2020.

[14] G. W. Stewart and J.-g. Sun, *Matrix Perturbation Theory*. Boston: Academic Press, 1990.

[15] P. Henrici, *Applied and Computational Complex Analysis*, vol. 1. New York: John Wiley & Sons, 1974.

[16] L. E. J. Brouwer, "Über abbildungen von mannigfaltigkeiten," *Mathematische Annalen*, vol. 71, pp. 97–115, 1911.

[17] E. Kreyszig, *Introductory Functional Analysis with Applications*. John Wiley & Sons, 1978.

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